Refresher Course on General Relativity
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Dr. Anuj Mishra

Tutorial 1: Classical Tests of General Relativity

In this tutorial we will work in natural units, where $G = c = 1$.

1. **Gravitational Frequency Shift and the Inevitable connection between Gravity and Spacetime**

   (a) Consider a stationary spherically-symmetric spacetime in the vicinity of a spherical mass M (e.g., a Schwarzschild spacetime). Consider an emitter at fixed spatial coordinates $(r_\mathrm{e},\ \theta_\mathrm{e},\ \phi_\mathrm{e})$ which emits a photon of frequency ν_e that is received by an observer at fixed spatial coordinates $(r_\mathrm{r},\ \theta_\mathrm{r},\ \phi_\mathrm{r})$ with frequency ν_r. Show that ν_E and ν_r are related by the expression:

   $$\frac{\nu_\mathrm{r}}{\nu_\mathrm{e}} = \left|\frac{g_{00}(r_\mathrm{e})}{g_{00}(r_\mathrm{r})}\right|^{1/2} \tag{1}$$

   (b) **Weak-Field Limit:**
   For Schwarzschild metric, show that in the weak-field limit ($\phi(r) = M/r \ll 1$), the above relation reduces to:

   $$\frac{\nu_\mathrm{r} - \nu_\mathrm{e}}{\nu_\mathrm{e}} \equiv \frac{\Delta\nu}{\nu_\mathrm{e}} = M\left(\frac{1}{r_\mathrm{r}} - \frac{1}{r_\mathrm{e}}\right) = \phi(r_\mathrm{r}) - \phi(r_\mathrm{e}) \tag{2}$$

   (c) **Pound-Rebka Experiment:**
   In 1960, Pound and Rebka experimentally proved the frequency shift of photons due to gravity by using a gamma ray from a 14.4 keV atomic transition in ^{57}Fe and making it fall vertically in the Earth's gravitational field through a distance of 22.6 meters. The experimental result was $\Delta\nu/\nu = (2.57 \pm 0.26) \times 10^{-15}$. Determine how well it agrees with the theoretical prediction. [Use: $g \approx 9.8ms^{-2},\ c \approx 3 \times 10^8 ms^{-1}$]

   (d) **Schild's Argument:**
   Discuss why the existence of gravitational frequency shift requires that spacetime must be non-Euclidean, leading naturally to the idea of spacetime curvature in general relativity.

   (e) **Gravitational Frequency Shift for General Spacetime and Observers:**
   The derivations presented above depend crucially upon the fact that the emitter and receiver are spatially fixed. However, this is not often physically realistic. Consider a general spacetime with metric $g_{\mu\nu}$ in some arbitrary coordinate system x^μ. Suppose that an emitter $\mathcal{E}$ and a receiver $\mathcal{R}$ have worldlines $x^\mu_\mathrm{e}(\tau_\mathrm{e})$ and $x^\mu_\mathrm{r}(\tau_\mathrm{r})$ respectively, where τ_e and τ_r are the proper times of each observer. At some event A, $\mathcal{E}$ emits a photon with 4-momentum $\boldsymbol{p}(A)$ that is received by $\mathcal{R}$ at an event B with momentum $\boldsymbol{p}(B)$. Furthermore, let us assume that at event A the emitter $\mathcal{E}$ has 4-velocity $\boldsymbol{u}_\mathrm{e}(A)$ and that at event B the receiver has 4-velocity $\boldsymbol{u}_\mathrm{e}(B)$. Show that:

   $$\frac{\nu_\mathrm{r}}{\nu_\mathrm{e}} = \frac{p_\mu(B)u^\mu_\mathrm{r}(B)}{p_\mu(A)u^\mu_\mathrm{r}(A)}, \tag{3}$$

   which for fixed observers, reduces to:

   $$\frac{\nu_\mathrm{r}}{\nu_\mathrm{e}} = \frac{p_\mu(B)}{p_\mu(A)}\left|\frac{g_{00}(A)}{g_{00}(B)}\right|^{1/2}. \tag{4}$$

2. **Shapiro Time Delay**
   The standard Schwarzschild metric in Schwarzschild coordinates (t, r, θ, ϕ) is given by:

   $$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),\ \text{where } f(r) = 1 - \frac{2M}{r}. \tag{5}$$

(a) **Isotropic Coordinate Transformation:**
By doing a coordinate transformation from r to ρ, defined by the relation

$$r = \rho \left(1 + \frac{M}{2\rho}\right)^2, \tag{6}$$

which is strictly monotonic for $r > 2M(\rho > M/2)$, transform the Schwarzschild metric to the following 'isotropic' form (which makes the coordinate speed of light direction-independent):

$$ds^2 = -\frac{(1 - M/2\rho)^2}{(1 + M/2\rho)^2}\, dt^2 + (1 + M/2\rho)^4 (dx^2 + dy^2 + dz^2). \tag{7}$$

(b) **Refractive Index of the Gravitational Field:**
Show that, in the new coordinates, the metric has an effective refractive index $n(\boldsymbol{x})$, where

$$\frac{d|\boldsymbol{x}|}{dt} = \frac{1}{n(\boldsymbol{x})}, \quad \text{and} \quad n(\boldsymbol{x}) = \frac{(1 + M/2|\boldsymbol{x}|)^3}{(1 - M/2|\boldsymbol{x}|)}. \tag{8}$$

(c) **Derivation for Shapiro Time Delay:**
The slowing down of light rays due to this 'refractive index' has been verified experimentally. Irwin Shapiro proposed this gravitational time delay (now called the *Shapiro time delay*) as the fourth test of general relativity. He suggested measuring the delay in radar signals transmitted between Earth and the inner planets (Venus or Mercury) when they are near superior conjunction (i.e., on the opposite side of the Sun from Earth). Show that if light travels from Earth at a point $\mathcal{P}_1$ located at radius ρ_1 from Sun to another point $\mathcal{P}_2$ located at radius ρ_2 from Sun, the one-way time delay compared to the scenario had there been no Sun is given by

$$\Delta t = \frac{2GM}{c^3} \ln\left(\frac{4\rho_1\rho_2}{b^2}\right) \tag{9}$$

where b is the impact parameter (the closest approach distance to the mass M). [*Hint:* Compute the line integral ndl along a straight line path in isotropic coordinates and assume $b \ll \rho_1, \rho_2$. Work in the weak-field limit.]

(d) **Order of magnitude of Shapiro time delay:**
Perform an order-of-magnitude estimation of the expected Shapiro time delay for such an experiment. Assume that the radio pulse grazes the surface of the Sun. [Use the following approximate values: $\rho_1 \sim \rho_2 \sim 1$ A.U. $\approx 10^{11}$ m; Solar mass: $M_\odot \approx 2 \times 10^{30}$ Kgs; Solar radius: $R_\odot \approx 7 \times 10^8$ m]

3. **Bending of light:**

(a) **Equation for a photon's trajectory:**
Show that the radial-coordinate r of a photon traversing through the Schwarzschild spacetime in the equatorial plane $\theta = \pi/2$ satisfies the equation:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}, \tag{10}$$

where λ is some affine parameter along the photon's trajectory, E is the conserved energy, and L is the conserved angular momentum of the photon.

(b) **Shape equation of a photon's orbit:**
To determine the shape of an orbit, we aim to write an equation relating r and ϕ, while eliminating the time dependence. Show that the shape of a photon's orbit can be expressed as:

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2, \tag{11}$$

where $u \equiv 1/r$. [Hint: First, obtain an equation for $(du/d\phi)^2$, then further differentiate it with respect to ϕ to obtain the desired expression.]

(c) **Photon's trajectory in absence of any matter:**
Show that in absence of any matter, the solution to the equation above can be expressed as $u_0 = \cos\phi/b$, where b is the *impact parameter* (distance of closest approach to origin of the coordinates).

(d) **Bending of light in the weak-field limit:**
Consider a photon coming from infinity and getting deflected due to the presence of a star of mass M as it goes back to infinity. In the weak-field limit ($M/r \ll 1$), one can assume that the shape of the trajectory is nearly straight. Thus, use an ansatz of the form $u(\phi) = u_0(\phi) + \delta u(\phi)$ in Eq. 11 and show that, up to first-order, $\delta u(\phi)$ satisfies:

$$\frac{d^2 \delta u}{d\phi^2} + \delta u = \frac{3M \cos^2 \phi}{b^2}, \tag{12}$$

having solution:

$$\delta u(\phi) = \frac{3M}{2b^2}\left(1 - \frac{1}{3}\cos 2\phi\right) \tag{13}$$

Taking the limit $r \to \infty$ ($u \to 0$), and $\phi \to \pm(\pi/2 + \delta_\phi)$, show that the total deflection of light $\Delta\phi$ is given by:

$$\Delta\phi = 2\delta_\phi = \frac{4GM}{bc^2} \tag{14}$$

(e) **Eddington's 1919 Eclipse Expedition:**
In 1919, Sir Arthur Eddington, a prominent British astrophysicist, went on an eclipse expedition to observe the eclipse from Príncipe, an island off the coast of West Africa, to test Einstein's theory of general relativity. The 1919 eclipse was chosen because it would take place with the Sun in front of a bright group of stars called the Hyades star cluster, which allowed Eddington to measure the deflection of starlight as it passed near the Sun. His team measured the deflection value to be $\Delta\phi = 1.61 \pm 0.4$ arcseconds. Compare this with the predicted value, assuming the light ray grazed the surface of the sun. [Use: Solar mass: $M_\odot \approx 2 \times 10^{30}$ Kgs; Solar radius: $R_\odot \approx 7 \times 10^8\ m$; speed of light: $c \approx 3 \times 10^8\ ms^{-1}$.]

4. **Precession of planetary orbits:**

(a) **Equation for a timelike particle's trajectory:**
Show that the radial-coordinate r of a timelike particle traversing through the Schwarzschild spacetime in the equatorial plane $\theta = \pi/2$ satisfies the equation:

$$\left(\frac{dr}{d\lambda}\right)^2 = \widetilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\widetilde{L}^2}{r^2}\right), \tag{15}$$

where λ is some affine parameter along the particle's trajectory, $\widetilde{E}$ is the conserved specific energy (E/m), and $\widetilde{L}$ is the conserved specific angular momentum of the photon (L/m).

(b) **Shape equation for timelike particles:**
Show that the shape of timelike particle can be expressed as:

$$\frac{d^2 u}{d\phi^2} + u = \frac{M}{\widetilde{L}^2} + 3Mu^2, \tag{16}$$

where $u \equiv 1/r$. Note that in the above, the first term on right hand side is what we get in Newtonian gravity while the second term is the leading order correction from Einstein's general relativity (GR). Also, note that this correction is of order v^2/c^2.

(c) **Shape equation in absence of correction from General Relativity:**
Ignore the correction coming from GR and show that the solution to the resultant equation can be expressed as:

$$u_0 = \frac{M}{\widetilde{L}^2}(1 + e\cos\phi), \tag{17}$$

where e is the *ellipticity* of the orbit.

(d) **Precession of planetary orbits for nearly circular orbits:**
For nearly circular orbits, use an ansatz of the form $u(\phi) = u_0(\phi) + \delta u(\phi)$ in Eq. 16 and show that, up to first-order, $\delta u(\phi)$ satisfies:

$$\frac{d^2 \delta u}{d\phi^2} + \delta u = \frac{3M^3}{L^4}(1 + e^2\cos^2\phi + 2e\cos\phi). \tag{18}$$

Solve this equation to get:

$$\delta u = \frac{3M^3}{L^4}\left(1 + e^2\left(\frac{1}{2} - \frac{\cos 2\phi}{6}\right) + e\phi\sin\phi\right) \tag{19}$$

Assuming $M^3/L^4 \ll 1$ (relativistic corrections to be small) and $e \ll 1$ (for nearly circular orbits), the above equation reduces to:

$$\delta u = \frac{3M^3}{L^4} e\phi\sin\phi \tag{20}$$

Using this, show that the final solution to Eq. 16 under these assumptions reduces to:

$$u(\phi) = \frac{M}{L^2}\left[1 + e\cos\left\{\phi(1-\alpha)\right\}\right], \tag{21}$$

where $\alpha = 3M^2/L^2$. Using this, show that the perihelion precession of planetary orbits is given by:

$$\Delta\phi = 2\pi\alpha = \frac{6\pi M^2}{L^2} \tag{22}$$

If a denotes the semi-major axis of the orbit, show that the above expression becomes:

$$\Delta\phi = \frac{6\pi GM}{a(1-e^2)c^2} \tag{23}$$

(e) The total observed precession of Mercury is $(5600.73 \pm 0.41)''$ per century, of which $(5557.62 \pm 0.4)''$ is due to the presence of other massive bodies in our solar system (of which Venus, Earth and Jupyter give the maximum contribution). The discrepancy between the two is $\sim 43.11''$, which was unaccounted for and had remained a mystery among astronomers untill Einstein showed in 1915 that GR perfectly explains this remaining precession. Using mercury's semi-major axis $a = 5.8 \times 10^8$ m, $e = 0.21$, $M_\odot \sim 2 \times 10^{30}$ kgs, and mercury's period to be 88 Earth days, show that GR indeed agrees well with the missing piece.